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LETTER TO THE EDITOR

Non-Abelian pseudopotentials and conservation laws of reaction-diffusion equations

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Abstract. We apply the Wahlquist–Estabrook technique to reaction-diffusion equations. A representation for non-Abelian pseudopotentials is obtained. An infinite hierarchy of conservation laws is derived from these pseudopotentials. The class of equations associated with such pseudopotentials is determined and examples are presented.

In the theory of nonlinear heat conduction and gas filtration through a porous medium (a review may be found in [1]) the nonlinear evolution equation of the second order:

$$u_t = (u^2)_{xx} + \alpha_1 u - \alpha_2 u^2 \qquad \alpha_1, \alpha_2 \neq 0 \tag{1}$$

arises. Here $u(x, t)$ denotes the temperature or gas density as a function of the time t and the space coordinate x.

By the following transformations

$$t \to t/\alpha_1$$

$$u \to u\alpha_1/\alpha_2$$

$$x \to \pm x/\sqrt{\alpha_2}$$

equation (1) is reduced to the canonical form

$$u_t = (u^2)_{xx} + u - u^2. \tag{2}$$

Equation (2) is 'non-integrable' and cannot be solved by means of the inverse scattering transformation. However, it has some particular solutions obtained and studied in [1].

In the present letter an infinite set of non-local conservation laws and two local conservation laws associated with them are obtained by employing the approach analogous to the method introduced in [2] and used in [3] for integrable equations. This approach is based on the existence of pseudopotentials of some kind for equation (2).

Later in this letter a problem on the presence of simplest pseudopotentials is considered, and explicit expressions for the pseudopotentials as well as the associated non-Abelian Lie algebra are obtained. We then present the derivation of conservation laws. The most common class of evolution equations associated with pseudopotentials of that special form has been considered and examples of such equations, of interest for physical models, are shown. The results and their connections with those presented earlier in [2] and [3] are finally discussed.

By means of the technique presented in [4] and [5] the general form of a pseudopotential for equation (2) can be found. Here we confine ourselves to so-called 'pseudopotentials of the first kind' according to the classification of [4]. Let us consider the pair of equations (vectorial in general) defining the pseudopotential $\boldsymbol{q}$

$$\begin{aligned} \boldsymbol{q}_x &= \boldsymbol{A}(u, \boldsymbol{q}) \\ \boldsymbol{q}_t &= \boldsymbol{B}(u, u_x, \boldsymbol{q}). \end{aligned} \tag{3}$$

For (3) to be integrable it is sufficient that

$$\boldsymbol{q}_{xt} - \boldsymbol{q}_{tx} = 0. \tag{4}$$

Assume that $u(x, t)$ is a solution of (2). Then (4) can be replaced by

$$\frac{\partial \boldsymbol{A}}{\partial u}((u^2)_{xx} + u - u^2) + [\boldsymbol{A}, \boldsymbol{B}] - \frac{\partial \boldsymbol{B}}{\partial u} u_x - \frac{\partial \boldsymbol{B}}{\partial u_x} u_{xx} = 0 \tag{5}$$

where

$$[\boldsymbol{A}, \boldsymbol{B}] = \boldsymbol{B}\frac{\partial \boldsymbol{A}}{\partial \boldsymbol{q}} - \boldsymbol{A}\frac{\partial \boldsymbol{B}}{\partial \boldsymbol{q}}.$$

Solving the equations for $\boldsymbol{A}$ and $\boldsymbol{B}$ at the higher-order derivatives in (3) as shown in [5], we get the general form of $\boldsymbol{A}$ and $\boldsymbol{B}$:

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{\alpha} u + \boldsymbol{\beta} \\ \boldsymbol{B} &= (u^2)_x \boldsymbol{\alpha} + u^2[\boldsymbol{\beta}, \boldsymbol{\alpha}] + \boldsymbol{\gamma} \end{aligned} \tag{6}$$

where $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ are vector-valued functions which depend only on $\boldsymbol{q}$ and satisfy the additional conditions

$$\begin{aligned} &[\boldsymbol{\alpha}, [\boldsymbol{\beta}, \boldsymbol{\alpha}]] = 0 \\ &[\boldsymbol{\beta}, [\boldsymbol{\beta}, \boldsymbol{\alpha}]] - \boldsymbol{\alpha} = 0 \\ &[\boldsymbol{\gamma}, \boldsymbol{\alpha}] - \boldsymbol{\alpha} = 0 \\ &[\boldsymbol{\gamma}, \boldsymbol{\beta}] = 0. \end{aligned} \tag{7}$$

As shown in [6], relations (7) are solvable only when they are consistent with some Lie algebra, where commutators in (7) correspond to multiplication.

Let $[\boldsymbol{\beta}, \boldsymbol{\alpha}] = \pm\boldsymbol{\alpha}$ and $\boldsymbol{\gamma} = \pm\boldsymbol{\beta}$ accordingly. Then (7) is satisfied identically. Moreover, it is clear that the Jacobi identities are also satisfied. Thus, elements $\{\boldsymbol{\alpha}, \boldsymbol{\beta}\}$ form the basis of a Lie algebra with multiplication table

$$\begin{aligned} &[\boldsymbol{\alpha}, \boldsymbol{\alpha}] = 0 \\ &[\boldsymbol{\beta}, \boldsymbol{\beta}] = 0 \\ &[\boldsymbol{\beta}, \boldsymbol{\alpha}] = \pm\boldsymbol{\alpha}. \end{aligned} \tag{8}$$

By definition, this algebra is non-Abelian. Finally, from (6) we have

$$\begin{aligned} \boldsymbol{a}_x &= \boldsymbol{\alpha} u + \boldsymbol{\beta} \\ \boldsymbol{q}_t &= ((u^2)_x \pm u^2)\boldsymbol{\alpha} \pm \boldsymbol{\beta}. \end{aligned} \tag{9}$$

Consider the linear pseudopotentials determined by formulae (8) and (9), i.e. take $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ such that

$$\boldsymbol{A} = P(u)\boldsymbol{q}$$
$$\boldsymbol{B} = Q(u, u_x)\boldsymbol{q}$$

where P and Q are $N \times N$ matrices. Then (2) may be presented in the form of the zero-curvature equation

$$\frac{\partial}{\partial t}P - \frac{\partial}{\partial x}Q + [P, Q] = 0 \tag{10}$$

where [,] denotes the matrix commutator.

It is obvious that $\boldsymbol{\alpha} = \alpha\boldsymbol{q}$ and $\boldsymbol{\beta} = \beta\boldsymbol{q}$, where α, β are matrices of the same dimensions as P and Q and

$$[\beta, \alpha] = \xi\alpha \qquad \xi = \pm 1. \tag{11}$$

The matrices P and Q are given by the formulae

$$\begin{aligned} P &= \alpha u + \beta \\ Q &= ((u^2)_x + \xi u^2)\alpha + \xi\beta. \end{aligned} \tag{12}$$

In the case dim $\alpha = 2$ the matrix equation (11) has the general solution (in the basis in which α has Jordan's normal form)

$$\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} \varepsilon & C \\ 0 & \varepsilon - \xi \end{pmatrix}.$$

Here C and ε are constants (in the following $C = 0$).

Let us write down equations (9) in terms of the components of vector $\boldsymbol{q} = (\omega_1, \omega_2)^{\mathrm{T}}$

$$\begin{aligned} \begin{pmatrix} \omega_{1x} \\ \omega_{2x} \end{pmatrix} &= \begin{pmatrix} \varepsilon\omega_1 + u\omega_2 \\ (\varepsilon - \xi)\omega_2 \end{pmatrix} \\ \begin{pmatrix} \omega_{1t} \\ \omega_{2t} \end{pmatrix} &= \begin{pmatrix} \varepsilon\xi\omega_1 + ((u^2)_x + \xi u^2)\omega_2 \\ (\varepsilon - \xi)\xi\omega_2 \end{pmatrix}. \end{aligned} \tag{13}$$

The system (13) can be solved consistently for ω_1 and ω_2. For ω_2 we have

$$\omega_{2x} = (\varepsilon - \xi)\omega_2$$
$$\omega_{2t} = (\varepsilon - \xi)\xi\omega_2.$$

ω_1 is the solution of the following system of equations

$$\begin{aligned} \omega_{1x} &= \varepsilon\omega_1 + u\omega_2 \\ \omega_{1t} &= \varepsilon\xi\omega_1 + ((u^2)_x + \xi u^2)\omega_2. \end{aligned} \tag{14}$$

The solution for ω_2 is easily determined within an arbitrary phase factor

$$\omega_2 = \exp((\varepsilon - \xi)(x + \xi t)).$$

We suppose that

$$\omega_1 = W_0 + \varepsilon W_1 + \varepsilon W_2^2 + \ldots$$
$$\omega_2 = \exp(-\xi(x + \varepsilon t))\left(1 + \sum_{i=1}^{+\infty} \frac{(x + \xi t)^i}{i!}\varepsilon^i\right).$$

It follows from (14) that W_i is defined by

$$(W_i)_x = W_{i-1} + u\exp(-\xi(x+\xi t))\frac{(x+\xi t)^i}{i!}$$

$$(W_i)_t = \xi W_{i-1} + ((u^2)_x + \xi u^2)\exp(-\xi(x+\xi t))\frac{(x+\xi t)^i}{i!} \tag{15}$$

$$i = 0, +\infty \qquad W_{-1} = 0 \qquad \xi = \pm 1.$$

In particular, for W_0 we have

$$(W_0)_x = u\exp(-\xi(x+\xi t))$$

$$(W_0)_t = ((u^2)_x + \xi u^2)\exp(-\xi(x+\xi t)) \qquad \xi = \pm 1.$$

Equations (15) determine two sets of potentials W_i for $\xi = \pm 1$ and, generally speaking, every potential function corresponds to a particular conservation law

$$\frac{\partial}{\partial t}(W_i)_x = \frac{\partial}{\partial x}(W_i)_t.$$

All conservation laws, except those corresponding to W_0 are non-local and all of them depend explicitly on the independent variables x and t. The first non-local conservation law in every set fits the potential with $i = 1$.

We now find a general form of the evolution equation associated with a pseudopotential of the type (9) and the Lie algebra determined by (8).

For this purpose we consider the equations defining the pseudopotential $\boldsymbol{q}$:

$$\begin{aligned} \boldsymbol{q}_x &= \boldsymbol{\alpha} u + \boldsymbol{\beta} \\ \boldsymbol{q}_t &= f(x,t)\boldsymbol{\alpha} + T(x,t)\boldsymbol{\beta}. \end{aligned} \tag{16}$$

In this case the integrability condition $\boldsymbol{q}_{xt} - \boldsymbol{q}_{tx} = 0$ will be written as

$$\boldsymbol{\alpha} u_t + uT[\boldsymbol{\alpha},\boldsymbol{\beta}] + f[\boldsymbol{\beta},\boldsymbol{\alpha}] - f_x\boldsymbol{\alpha} - T_x\boldsymbol{\beta} = 0.$$

We assume that

$$[\boldsymbol{\beta},\boldsymbol{\alpha}] = \nu\boldsymbol{\alpha} \qquad \nu = \text{constant}. \tag{17}$$

Then we have

$$\boldsymbol{\alpha} u_t - \nu u T\boldsymbol{\alpha} + \nu f\boldsymbol{\alpha} - f_x\boldsymbol{\alpha} - T_x\boldsymbol{\beta} = 0.$$

Equating the coefficients at a and b to zero, we get

$$u_t - f_x + \nu f - \nu u T = 0$$

$$T_x = 0$$

or

$$\begin{aligned} &u_t - f_x + \nu f - \nu u T(t) = 0 \\ &f = f(x, t, u, u_x, \ldots, u_{nx}). \end{aligned} \tag{18}$$

The following are some equations presented earlier in works of other authors and occurring in various physical problems.

(*a*) An equation analogous to the well known Korteweg–de Vries–Burgers equation was considered in [7] in connection with the classification of nonlinear equations of the form

$$u_t = \tfrac{1}{2}(u^2)_{xx} + \tfrac{1}{2}(u^2)_x + u_{xx} + u_{xxx}$$

without an infinite set of local conservation laws. The equation can be represented in the form (18) for the following range of parameters

$$\nu = -1$$

$$T = 0$$

$$f = \tfrac{1}{2}(u^2)_x + u_{xx}.$$

(*b*) A particular form of the so-called generalized Burgers equation [8, 9]

$$u_t = -u^\beta u_x - \lambda u^\alpha + \frac{\delta}{2} u_{xx} \qquad \lambda \neq 0$$

in the case $\beta = \alpha - 1$ and

$$\nu = -\lambda\alpha$$

$$T = -\frac{\lambda\alpha\delta}{2}$$

$$f = \frac{-u^\alpha}{\alpha} + \frac{\delta}{2} u_{xx} - \frac{\lambda\alpha\delta}{2} u.$$

(*c*) The generalized Fisher equation [10, 11]

$$u_t + \alpha u u_x - u_{xx} = \beta u(1-u) \qquad \alpha, \beta \neq 0$$

$$\nu = -\frac{2\beta}{\alpha}$$

$$T = -(\alpha^2 + 4\beta)/(2\alpha)$$

$$f = -\frac{\alpha}{2} u^2 + u_x - \frac{2\beta}{\alpha} u.$$

For each of these equations an infinite set of non-local conservation laws analogous to (15) can be derived by means of the technique presented earlier.

In the present work we have used a method of pseudopotentials of 'prolonged structures' to obtain infinite sets of non-local conservation laws. Moreover, the class of evolution equations associated with pseudopotentials analogous to the pseudopotentials used for equation (2) has been determined. For such equations, sets of non-local conservation laws can be obtained. Examples of these 'non-integrable' equations, interesting from the viewpoint of physical problems, have also been presented. The technique described above is similar to the method proposed in [2] and [3] for integrable equations and based on the use of a special type of relations. Such relations arise in the theory of pseudospherical surface (PSS), which are by definition one-dimensional non-Abelian pseudopotentials of the first kind. We believe the procedure for constructing non-local conservation laws from non-Abelian pseudopotentials, which is not associated with presentation of the differential equation in a form describing a PSS, is more straightforward and can be used for 'non-integrable' equations.

It is essential to note that we used a two-dimensional pseudopotential of the simplest type (13). The one-dimensional equation (14) finally obtained for the pseudopotential belongs to the more complicated class of equations with coefficients depending on x and t. The problem of finding one-dimensional pseudopotentials leading to (14) is very difficult [6].

Note that an infinite set of conservation laws results from invariance under the transformation

$$P \to \bar{\bar{P}} \equiv P + \varepsilon I$$
$$Q \to \bar{\bar{Q}} \equiv Q \mp \varepsilon I$$
$$I = \text{matrix unit.}$$

This is so because relations (12) can be written as

$$P = \begin{pmatrix} 0 & u \\ 0 & \pm 1 \end{pmatrix} + \varepsilon I$$
$$Q = \begin{pmatrix} 0 & (u^2)_x \pm u^2 \\ 0 & -1 \end{pmatrix} \pm \varepsilon I.$$

Pseudopotentials found previously can also be used for obtaining the Bäcklund transformations of (2) and so for other equations converted to the class (18) and subsequently considered. For example, equation (4) provides the Bäcklund transformation ($\varepsilon = 0$)

$$u(x, t) = \omega_x \exp(-\xi(x + \xi t)) \qquad \xi = \pm 1$$

from equation

$$\omega_t = (3\xi\omega_x^2 + (\omega_x^2)_x) \exp(-\xi(x + \xi t))$$

into equation (2).

At present we study the question of employing the Bäcklund transformations to obtain the exact solutions of equation (2) and equations (6)-(18).

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